Summary of my paper entitled

"My Challenge on the Development of a Mixed Variational Formulation in Solid Mechanics"

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## 1. Historical survey of the mixed variational formulation

In 1950 Eric Reissner published a paper on a new variational formulation of the boundary value problems of solid mechanics which opened the door to the mixed method of solution on displacements $u_i$ and stresses $\sigma_{ij}$.

In March 1955 H. C. Hu and K. Washizu succeeded in development of the most general mixed variational principle using Lagrange multiplier from the minimum principle of potential energy.

Washizu discussed that all existing variational principles are mutually connected and transformed each other by Friedrichs transformation from the principle of virtual work to the principle of complemental virtual work vice versa, via Reissner's principle.

Their works gave great impact to further development of the finite element method up to the present.

## 2. Motivation of my research

To develop his own principle, E. Reissner employed Lagrange multiplier method to introduce the displacement boundary condition to the principle of minimum potential energy.

Consequently, convergency of approximate solutions to the true solution can not be guaranteed.

Therefore development of a new mixed variational principle without using Lagrange multiplier has become motivation of my research on the finite element method since 1956 when Jon Turner's Boeing research group published a paper on the direct stiffness method of solution and decline of the Force Method started.

## 3. Outline of my variational formulation

Firstly I paid attention on the well-known energy conservation law in mechanics as given by

$$\int_v \sigma_{ij}\varepsilon_{ij}dV = \int_v \overline{p}_i u_i dV + \int_{S_\sigma} \overline{t}_i u_i dS + \int_{S_u} \overline{u}_i t_i dS \qquad \text{.........................................(3.1)}$$

where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\sigma_{ij} = \sigma_{ji}$ are assumed, and they are in equilibrium with the body force $\overline{p}_i$ $in\, V$, surface traction $\overline{t}_i$ on $S_\sigma$ and enforced displacement $\overline{u}_i$ on $S_u$, and

$$\sigma_{ij,i} + \overline{p}_i = 0 \text{ in V}, \ t_i = \overline{t}_i \text{ on } S_\sigma, \ t_i = \sigma_{ij} n_j \qquad \text{..................................... (3.2)}$$

$n_j$ is an unit normal drawn outward on the stress prescribed boundary $S_\sigma$ and $u_i = \overline{u}_i$ on the displacement prescribed boundary $S_u$ where $S = S_\sigma + S_u$

Eq.(3.1) is valid under isothermal or adiabatic temperature conditions.

Hereafter standard notations in mechanics will be introduced without detailed explanation.

Eq.(3.1) states "when a solid is deformed and in equilibrium with external body force $\overline{p}_i$ and surface traction $\bar{t}_i$ on $S_\sigma$ plus enforced displacement $\overline{u}_i$ on $S_u$, the strain energy is equal to work done due to these external forces and displacement."

In practice, we must be satisfied with the approximate solution for the state vector $(u_i, \sigma_{ij})$ of a given solid.

Therefore taking variations with respect to $u_i$ .and $\sigma_{ij}$, the following equation can be derived.

$$\delta\int_v \sigma_{ij}\varepsilon_{ij} dV - \int_v \overline{p}_i \delta u_i dV - \int_{S_\sigma} \bar{t}_i \delta u_i dS - \int_{S_u} \overline{u}_i \delta t_i dS = 0 \qquad \text{.................................... (3.3)}$$

$$(w.r.t.\ u_i \ \&\ \sigma_{ij})$$

It must be necessary to introduce the stress-strain law of a solid under consideration to use eq.(3.3) as shown in Fig.1.

Strain energy $A(\varepsilon)$:

$$A(\varepsilon) = \int_c \sigma d\varepsilon \qquad \text{.....................(a)}$$

where c is the loading path in $\sigma - \varepsilon$ space.

Complementary strain energy $B(\sigma)$

$$B(\sigma) = \int_c \varepsilon d\sigma \qquad \text{.....................(b)}$$

$$A(\varepsilon) + B(\sigma) = \sigma\varepsilon \qquad \text{.....................(c)}$$

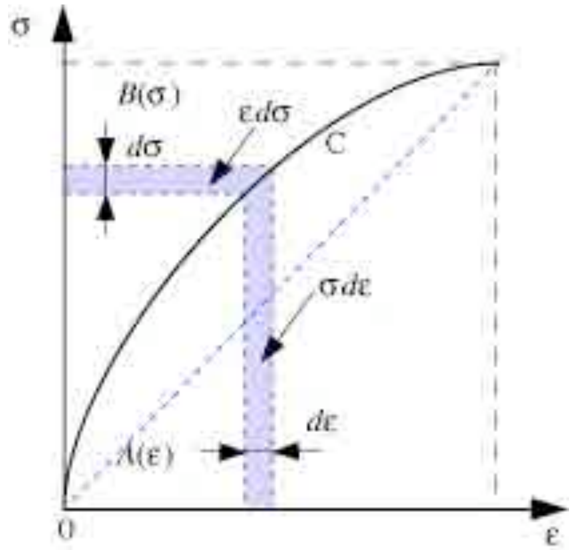



Fig. 1 Stress-strain law and definition of the strain energy $A(\varepsilon)$ and complementary strain energy $B(\sigma)$, c is loading path on $\sigma - \varepsilon$ diagram

Suppose that the stress-strain relation is given by

$$\sigma_{ij} = f(\varepsilon_{kl}) \quad i, j, k, l = 1,2,3 \qquad \text{....................... (3.4)}$$

as in the theory of total deformation or flow theory of plasticity.

If $\sigma_{ij} = 0$ corresponds to $\varepsilon_{kl} = 0$ and Jacobian $\partial(\sigma_{11}, \sigma_{22} \cdots)/\partial(\varepsilon_{11}, \varepsilon_{22} \cdots) \neq 0$ for the entire domain defined including $(\sigma_{ij} = 0, \varepsilon_{kl} = 0)$, then the inverse function of eq.(3.4) can be defined uniquely as follows;

$$\varepsilon_{ij} = g(\sigma_{kl}) \quad (i,k,k,l = 1,2,3) \qquad \text{.................................... (3.5)}$$

Furthermore existence of the following strain energy function $A(\varepsilon_{ij})$ and its positive–definiteness are assumed by

$$dA(\varepsilon_{ij}) = \sigma_{ij} d\varepsilon_{ij} \qquad \text{.................................... (3.6)}$$

$$\delta^2 A = \frac{\partial^2 A}{\partial \varepsilon_{ij} \varepsilon_{kl}} \delta\varepsilon_{ij} \delta\varepsilon_{kl} \geq 0 \quad \text{.....................................} (3.7)$$

Then the complementary energy function $B(\sigma_{ij})$ and its positive definiteness are also assured by

$$dB(\sigma_{ij}) = \varepsilon_{ij} d\sigma_{ij} \quad \text{.....................................} (3.8)$$

$$\delta^2 B = \frac{\partial^2 B}{\partial \sigma_{ij} \partial \sigma_{kl}} \delta\sigma_{ij} \delta\sigma_{kl} \geq 0 \quad \text{.....................................} (3.9)$$

Therefore if these conditions (3.4)-(3.9) are assumed, eq.(3.3) can be expressed by

$$(\int_v \delta A(\varepsilon_{ij}) dV - \int_v \bar{p}_i \delta u_i dV - \int_{S_\sigma} \bar{F}_i \delta u_i dS) + (\int_v \delta B(\sigma_{ij}) dV - \int_v \bar{u}_i \delta t_i dS) = 0 \quad \text{...........} (3.10)$$

$$(w.r.t.\ u_i)$$

The first and second parentheses of L. H. S. of eq.(3.10) equal to zero represent coexistence of principles of the virtual work $(w.r.t.\ u_i)$ and the complementary virtual work $(w.r.t.\ \sigma_{ij})$ respectively.

Therefore it can be concluded that eq.(3.3) is a new mixed variational principle which unifies principles of virtual work and complementary virtual work.

Especially in case of the linear elasticity, the stress-strain law is given by

$$\left.\begin{aligned} \sigma_{ij} &= a_{ijkl} \varepsilon_{kl} \\ or \quad \varepsilon_{ij} &= b_{ijkl} \sigma_{ij} \end{aligned}\right\} \quad \text{.....................................} (3.11)$$

Then, two equations for principles of virtual work and complementary virtual work become independent each other because

$$A(\varepsilon_{ij}) = B(\sigma_{ij}) = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \geq 0 \quad \text{.....................................} (3.12)$$

the former gives the upper bound solution while the latter gives the lower bound solution of the strain energy.

Thus it can be concluded that monotonously convergent approximate solutions can be always obtained using this mixed energy principle given by eq.(3.3) if the existence of $A(\varepsilon_{ij})$ or $B(\sigma_{ij})$ and their positive-definiteness are assured(See Fig.2).

4. <u>Comparison on the approximate solutions obtained by Reissner's principle and the present energy formulation</u>

Alternative form of eq.(3.3) can be given by;

$$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS+\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS-\int_v(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV-\int_v u_i\delta\sigma_{ij,j}dS=0 \qquad \text{.........(4.1)}$$

The last term of the *L.H.S* of eq.(4.1) is considered first.

The equilibrium equation of a solid is given by;

$$\sigma_{ij,j}+f_i=0 \quad \text{in V} \qquad \text{......................................(4..2)}$$

where $f_i$ is some distributed body force.

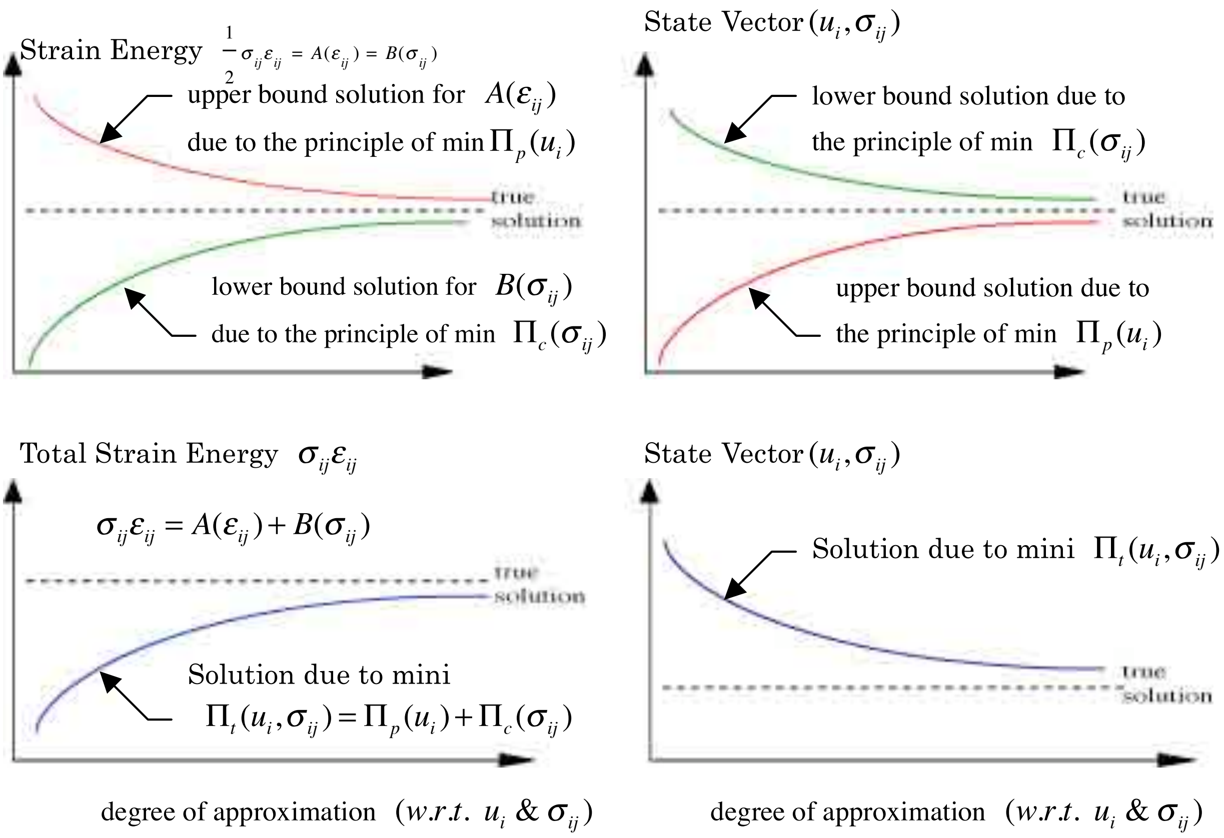
Typical body forces are the gravitational force, thermal load due to temperature distribution, and so on. Therefore in case of the pure mechanics problem, the last energy integral of eq.(4.1) can be deleted. The similar equation of eq.(4.1) can be derived in case of Reissner's principle as follows;

$$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS-\int_{S_\sigma}(u_i-\bar{u}_i)\delta t_i dS-\int_v(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0 \qquad \text{............................(4.3)}$$

Now difference of eq.(4.1) and eq.(4.3) is obvious, i.e. only difference is sign of the second term.

Therefore it can be concluded that monotonous convergency of approximate solutions for the state vector $(u_i,\sigma_{ij})$ can be expected in the new mixed method, while in case of the method based on Reissner's Principle such convergency is not guaranteed.

This conclusion is true irrespective of the stress-strain law if existence and positive definiteness of $A(\varepsilon_{ij})$ or $B(\sigma_{ij})$ are assured(See Fig. 2).



$$\Pi_p(u_i) = \int_v A(\varepsilon_{ij})dv - \int_v \overline{p}_i u_i dv - \int_{s_\sigma} \overline{t}_i u_i ds$$

$$\Pi_c(\sigma_{ij}) = \int_v B(\sigma_{ij})dv - \int_{s_u} \overline{u}_i t_i ds$$

$$\Pi_t(u_i, \sigma_{ij}) = \Pi_p(u_i) + \Pi_c(\sigma_{ij})$$

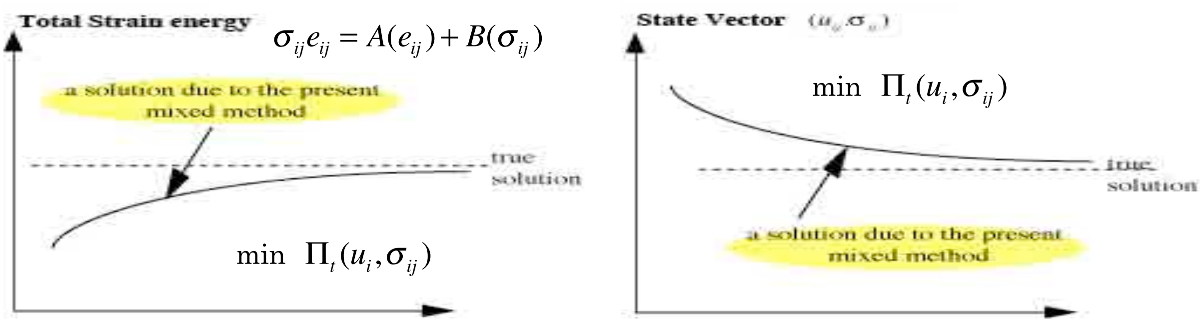
*If* $(u_{ij}, \sigma_{ij})$ *is the true solution*

$$\Pi_t(u_i, \sigma_{ij}) \rightarrow \min\ (w.r.t.\ u_i\ \&\ \sigma_{ij})$$

$\min \Pi_c(u_i, \sigma_{ij})$ ↔ $\min \Pi_p(u_i)$

$\min \Pi_c(u_i, \sigma_{ij})$ ↔ $\min \Pi_c(\sigma_{ij})$

(a) linear elasticity problem




degree of approximation $(w.r.t.\ u_i\ \&\ \sigma_{ij})$ degree of approximation $(w.r.t.\ u_i\ \&\ \sigma_{ij})$

(b) nonlinear elasticity problem

Fig 2. Convergency characteristic of approximation solutions to be obtained basing on the present mixed variational method

In case of nonlinear problems, decoupling of the unified energy principle is not possible, and therefore bracketing of the true solution in-between the upper and lower bound solutions can not be made, but approximate solution will monotonously converge to the true solution from upper side or lower side of the true solution definitely so long as existence of $A(\varepsilon_{ij})$ and $B(\sigma_{ij})$, together with their positive definiteness are assumed. It should be mentioned here that the total strain energy of system $\sigma_{ij}\varepsilon_{ij}$ is drawn in case of the conservative stiffness estimation for structural design.

5. <u>There are 8 possible methods of solution in the mixed finite element formulation</u>

In the mixed finite element analysis 8 different methods of solution can be proposed considering combination of the following three conditions to be satisfied in each individual element:

$$\sigma_{ij} + \overline{p}_i = 0 \quad in\, V\ ,\ \ t_i - \overline{t}_i = 0 \ \text{ on } \ S_\sigma \ \text{ and } \ u_i - \overline{u}_i \ \ on\, S_u \quad (w.r.t.\ each\ element).$$

where $(\overline{u}_i, \overline{t}_i)$ must be understood as unknown state vector of the adjacent elements on the same element boundary surface.

They are shown in Table 1. Solution (1) (tentatively called as "Modified Reissner Method") covers other 7 methods and Solution (5) is the well-known Trefftz's method by which the lower bound solution can be always obtained. These solution procedure ① and ⑤ are new unique methods where continuity of the element state vectors along their boundary surfaces are not required a priori

and the other 6 methods are, then, called as "Generalized Finite Element Method."
Among them, DM( I ) corresponds to the conventional finite element method. It is also interesting to mention that GM( II ) is a semi-analytical method because all three element conditions are a priori satisfied.

Table1: 8 Possible methods of solution derived from the present mixed variational formulation (for Rayleigh-Ritz's method)

| SOL NO. | Variational equations | Constraint Conditions | remarks |
|---|---|---|---|
| 1 | $\int_V (\sigma_{ij,j}+\bar{p}_i)\delta u_i dV - \int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS - \int_{S_u}(u_i-\bar{u}_i)\delta t_i dS = 0$ | ———— | modified Reissner's Method |
| 2 | $\int_V (\sigma_{ij,j}+\bar{p}_i)\delta u_i dV - \int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS = 0$ | $u_i-\bar{u}_i=0$ on $S_u$ | Displacement Method (I) (DM I) |
| 3 | $\int_V (\sigma_{ij,j}+\bar{p}_i)\delta u_i dV - \int_{S_u}(u_i-\bar{u}_i)\delta t_i dS = 0$ | $t_i-\bar{t}_i=0$ on $S_\sigma$ | Equilibrium Method (I) (EM I) |
| 4 | $\int_V (\sigma_{ij,j}+\bar{p}_i)\delta u_i dV = 0$ | $u_i-\bar{u}_i=0$ on $S_u$<br>$t_i-\bar{t}_i=0$ on $S_\sigma$ | Galerkin's Method (I) |
| 5 | $\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS + \int_{S_u}(u_i-\bar{u}_i)\delta t_i dS = 0$ | $\sigma_{ij,j}+\bar{p}_i=0$ in V | Trefftz's Method |
| 6 | $\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS = 0$ | $\sigma_{ij,j}+\bar{p}_i=0$ in V<br>$u_i-\bar{u}_i=0$ on $S_u$ | DM(II) |
| 7 | $\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS = 0$ | $\sigma_{ij,j}+\bar{p}_i=0$ in V<br>$t_i-\bar{t}_i=0$ on $S_\sigma$ | EM(II) |
| 8 | ———— | $\sigma_{ij,j}+\bar{p}_i=0$ in V<br>$t_i-\bar{t}_i=0$ on $S_\sigma$<br>$u_i-\bar{u}_i=0$ on $S_u$ | GM(II) Analytical Solution |

Theoretically FEM using GM(Ⅱ) elements is equivalent to the Boundary Element Method where the element characteristic matrix which consists of stiffness and flexibility matrices is analytical and therefore the matrix size of the overall system must be very small to compare with conventional finite element analysis and furthermore study of design parameters may be possible, although derivation of the element characteristic matrix might not be easy.

In the mixed FEM analysis solutions ① and ⑤ can be recommended for practical use, because preliminary fulfillment of continuity of the element state vectors on their common boundaries can be avoided in these methods of solution, so that finite element analysis may be greatly simplified.

6. Role of the stress-strain law on the development of the present mixed method of solution

The present mixed variational method cannot be useful unless the stress-strain law of a material used is specified either in the form of $\sigma_{ij} = f(\varepsilon_{ij})$ or $\varepsilon_{ij} = g(\sigma_{ij})$.

So that the strain energy function $A(\varepsilon_{ij})$ and its complementary strain energy function $B(\sigma_{ij})$ can be supplied at least in Maclaurin series form.

Prof. I. Shibuya of Osaka University gave the following formula for strain energy function $A(e_{ij})$ of the metal from the thermodynamics point of view:

$$A(e_{ij}) = A_0 + \frac{1}{2!} a_{ijkl} e_{ij} e_{kl} + \frac{1}{3!} a_{ijklmn} e_{ij} e_{kl} e_{mn} + \cdots \qquad \text{…………………(6.1)}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} \cdot u_{k,i})$ $(i, j, k = 1,2,3)$

Since $\sigma_{ij} = \frac{\partial A}{\partial e_{ij}} = f(e_{ij})$ and if Jacobian $\partial(\sigma_{11}, \sigma_{22} \cdots)/\partial(e_{11}, e_{22} \cdots) \neq 0$ the inverse of $\sigma_{ij} = f(e_{ij})$ i.e., $e_{ij} = g(\sigma_{ij})$ can be assured.

For practical use of this mixed variational formulation, however, derivation of $A(e_{ij})$ and $B(\sigma_{ij})$ in the polynomial form will play a vital role in its future development.

Concerning this point, W. Prager assumed first the most general nonlinear elasticity law of the following type;

$$e' = c_1 \sigma_1' + c_2 \sigma'^2 + c_3 \sigma'^3 + \cdots \qquad \text{…………………………………………(6.2)}$$

where superscript implies deviatoric stress or strain tensors.

And then by applying Cayley-Hamilton's Theorem, he reduced it to the following equation:

$$e' = P\sigma' + Q\sigma'^2 \qquad \text{…………………………………………(6.3)}$$

*(Ref. W. Prager; "On the Kinematics of Soils "Memories des Sciences 28(1954), Ac.Roy.Belgique pp.3-8)*

However, I recognize that this is only the first step of my work on applications of the mixed variational method to analysis of general nonlinear problems including metals, high polymers, rubbers, soil, rock, concrete, many other composites and bio-materials.

7. Development of nodeless finite element method

I think that epoch making progress of finite element method may be attributed to generalization of the joint concept in the frame analysis to continuum mechanics, but recently I recognize that this concept has been giving undesirable influence on the further development of nonlinear finite element analysis where the incremental method plays vital role of analysis.

In the conventional finite element analysis, equality of two functions $f(x) = g(x) \quad (a \le x \le b)$ is approximated by $f(x_i) = g(x_i) \quad (i = 1,2,\cdots n, a \le x_i \le b)$.

This concept is very practical and easy to apply.

However, it presents difficult problems in the solid contact problem where contact area as well as pressure distribution on the surface are a priori unknown and they may change depending upon the progress of loading.

Therefore I believe that the equation; $f(x_i) = g(x_i)$ should be replaced by the following equality equations of polynomials of finite degree;

$$f(x_i) = g(x_i) \rightarrow \sum_{k=1}^{u} a_k x^k = \sum_{k=1}^{u} b_k x^k \rightarrow a_k = b_k$$

$$(k = 0,1 \cdots u)$$

In this way, nodes can be completely eliminated in the finite element analysis. Application of the new mixed variational formulation is now under way to analysis of the solid contact problems.

8. Force method, its rise and fall

Force method was originally proposed in the frame analysis, but it has been declined because concept of the redundant forces is not justified in the frame analysis and in continuum mechanics.

However it is generally clear that external forces are in equilibrium in the deformed solids. Therefore the Force Method can be evidently restored as the Equilibrium Method imposing no rigid body displacement condition to the structure under consideration in the present mixed variational method. The conclusion will be illustrated by a simple space frame analysis.

Generally speaking, each one dimensional member(beam or column) has 12 degrees of freedom $\left(u, v, w, \theta, \phi, \chi; V_x, V_y, P, M_x, M_y, M_z\right)$ i.e. 6 displacements components and their corresponding forces, and therefore it is not difficult to construct the element characteristic matrix which consists of the stiffness and flexibility matrices of the beam element.

Collecting equilibrium and compatibility equation at all joints of a given frame in matrix form, the following total equation for state vectors of a given frame can be obtained in matrix form:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} d \\ f \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{......................................(8.1)}$$

where $d$ is nodal displacement, $f$ is the corresponding force vector.

$$\left.\begin{aligned} K_{11}d + K_{12}f = F_1 \\ K_{21}d + K_{22}f = F_2 \end{aligned}\right\} \quad \text{.......................................(8.2)}$$

Eq.(8.1) or eq.(8.2) are characteristic equations for state vectors of a given frame in the mixed form. It is easy to see that they consist of stiffness and flexibility equations of a given frame.

Thus, the force method is restored by matrix condensation technique and it will be illustrated by analysis of a simple space frame.

9. Castigliano's Theorem can be derived from the Energy Conservation Law

It is not too difficult to derive the well known Castigliano's Theorem from the energy conservation law given by eq.(9.1). To do this, it is only necessary to take changes of external body force $\overline{p}_i \quad in\, V$, surface tractions $\overline{t}_i \quad on\, S_\sigma$ and enforced displacements $\overline{u}_i \quad on\, S_u$.

$$\int_v \sigma_{ij}\varepsilon_{ij}dv = \int_v \overline{p}_i u_i dv + \int_{s_\sigma} \overline{t}_i u_i ds + \int_{s_u} \overline{u}_i t_i ds \quad \text{..............................(9.1)}$$

$$\begin{aligned} \int_v (\sigma_{ij} + \Delta\sigma_{ij})(\varepsilon_{ij} + \Delta\varepsilon_{ij})dv = \int_v (\overline{p}_i + \Delta\overline{p}_i)(u_i + \Delta u_i)dv \\ + \int_{s_\sigma} (\overline{t}_i + \Delta\overline{t}_i)(u_i + \Delta u_i)ds + \int_{s_u} (\overline{u}_i + \Delta\overline{u}_i)(t_i + \Delta t_i)ds \end{aligned} \quad \text{..............(9.2)}$$

where $\Delta$ must be applied to external forces $\overline{p}_i$ body force in V, $\overline{t}_i$ surface tractions on $S_\sigma$ and $\overline{u}_i$ enforced displacements on $S_u$.

Considering resulting small change of the state vector $(u_i, \sigma_{ij})$ i.e., $(\Delta u_i, \Delta\sigma_{ij})$ and neglecting the higher order terms the following equation can be derived:

$$\begin{aligned} (\int_v \sigma_{ij}\Delta\varepsilon_{ij}dv - \int_v \overline{p}_i \Delta u_i dv - \int_{s_\sigma} \overline{t}_i \Delta u_i ds - \int_{s_u} t_i \Delta\overline{u}_i ds) \\ + (\int_v \varepsilon_{ij}\Delta\sigma_{ij}dv - \int_v \Delta\overline{p}_i u_i dv - \int_{s_\sigma} \Delta\overline{t}_i u_i ds - \int_{s_u} \overline{u}_i \Delta t_i ds) = 0 \end{aligned} \quad \text{..............(9.3)}$$

The first equation of the L. H. S. of eq.(9.3) represents change of eq.(9.1) w.r.t. the enforced displacements $\overline{u}_i \quad on\, S_u$ and the second equation change of eq.(9.1) due to external force $(p_i, t_i)$ and they are considered independent. Thus the following two equations of so called the first and second theorem of Castigliano can be derived:

(i) First theorem of Castigliano

$$\int_v \Delta A\left(\varepsilon_{ij}\right)dv - \int_v \overline{p}_i \Delta u_i dv - \int_{s_\sigma} \overline{t}_i \Delta u_i ds - \int_{s_u} t_i \Delta\overline{u}_i ds = 0 \quad \text{..............(9.4)}$$

$$t_i = \frac{\partial \Pi_p}{\partial \overline{u}_i} \quad \text{..............(9.5)}$$

where $\Pi_p(u_i) = \int_v A\left(\varepsilon_{ij}\right)dv - \int_v \overline{p}_i u_i dv - \int_{s_\sigma} \overline{t}_i u_i ds$ ..............(9.6)

is the potential energy of a given system.

(ii) Second theorem of Castigliano

$$\int_v \Delta B\left(\sigma_{ij}\right)dv - \int_v \Delta\bar{p}_i u_i dv - \int_{s_\sigma} \Delta\bar{t}_i u_i ds - \int_{s_u} \Delta t_i \bar{u}_i ds = 0 \qquad \text{.............(9.7)}$$

$$u_i = \frac{\partial \Pi_c}{\partial \bar{t}_i} \qquad \text{.............(9.8)}$$

where $\Pi_c(u_i)$ is the complementary energy of a given system expressed by

$$\Pi_c(\sigma_{ij}) = \int_v B\left(\sigma_{ij}\right)dv - \int_v \bar{u}_i t_i ds \qquad \text{.............(9.9)}$$

and

$$\left.\begin{aligned} A(\varepsilon_{ij}) &= \int_c \sigma_{ij} d\varepsilon_{ij} \\ B(\sigma_{ij}) &= \int_c \varepsilon_{ij} d\sigma_{ij} \end{aligned}\right\} \qquad \text{.............(9.10)}$$

It is interesting to note that another theorem can be also derived from eq.(9.7) by:

$$u_i = \frac{\partial \Pi_c}{\partial \bar{p}_i} \qquad \text{.............(9.11)}$$

10. A new mixed variational formulation of solid mechanics problems in terms of deviatoric stress and strain tensors

In the theory of plasticity, it is common to use the following deviatoric stress and strain tensors:

$$\left.\begin{aligned} \sigma_{ij} &= \sigma'_{ij} + p\delta_{ij} \\ \varepsilon_{ij} &= \varepsilon'_{ij} + e\delta_{ij} \end{aligned}\right\} \qquad \text{.............................. (10.1)}$$

where $p = \frac{1}{3}\sigma_{kk}$ , $e = \frac{1}{3}\varepsilon_{kk}$

where $p$ is the hydrostatic pressure and $e$ is the corresponding volumetric strain.

And it is not difficult to derive the following equation on the strain energy expression:

$$\sigma_{ij}\varepsilon_{ij} = \sigma'_{ij}\varepsilon'_{ij} + pe \qquad \text{.............................. (10.2)}$$

That is to say, the strain energy of a solid consists of two parts, i.e. one is due to volumetric change of a solid, while the other, strain energy due to the shape change. It is common practice in the flow theory of plasticity to assume incompressibility of a solid and then $pe$ is neglected. In the slow viscous flow problem, it is also common to neglect this term.

However, incompressibility of a material under consideration is assumption by which the analysis may be simplified, but often it may give some trouble in analysis.

Therefore, original variational principle given by eq.(10.1) is slightly modified using eq.(10.1) and eq.(10.2) as follows:

$$\begin{aligned} &\delta\int_v \sigma'_{ij}\varepsilon'_{ij}dv + \delta\int_v pe dv - \int_v \bar{p}_i \delta u_i dv - \int_{s_\sigma} \bar{t}_i \delta u_i ds \\ &- \int_{s_u} \bar{u}_i \delta t'_i ds - \int_{s_u} \bar{u}_n \delta p ds = 0 \end{aligned} \qquad \text{.............................. (10.3)}$$

where $u_n$ is the enforced surface displacement along the normal direction $n$ on the surface.

$t_i = \sigma_{ij} n_j = (\sigma'_{ij} + p\delta_{ij}) n_j = t'_i + p \quad t'_i = \sigma'_{ij} n_j$

Increase of 2 unknown variables $p$ and $e$ is compensated by reduction of any 2 unknowns $\sigma'_{ij}$ and $\varepsilon'_{ij}$ from $\sigma'_{ij} = \varepsilon'_{ij} = 0$. Original formulation was first proposed by L. R. Herrmann* for analysis of the viscoelastic materials.

** L. R. Herrmann and R. M. Toms, "A reformulation of the Elastic Field Equations, in terms of Displacements, valid for all admissible values of Poisson's Ratio" Transaction of ASME, Journal of Applied Mechanics, vol. 86, Ser, E., pp. 140~141, 1964*

11. <u>Application of the new mixed variational formulation to the finite deformation problem of elastic solids</u>

Changing the definition of strains from $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ to $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} \bullet u_{k,j})$, this mixed variational method can be extended to analysis of the finite elastic deformation problems as follows;

$$\delta \int_v \sigma_{ij} e_{ij} dV - \int_v \bar{p}_i \delta u_i dV - \int_{S_\sigma} \bar{f}_i \delta u_i dS - \int_{S_u} \bar{u}_i \delta f_i dS = 0 \quad \text{.............................. (11.1)}$$

$$(w, r, t, u_i \ \& \ \sigma_{ij})$$

where $f_i = \sigma_{ij} u_k (\delta_{kj} + u_{k,i})(r, i, k = 1,2,3)$ $\bar{f}_i$; given surface tractions

It must be mentioned here that independent variational formulation $w.r.t.\ u_i \ \& \ \sigma_{ij}$ is no longer possible in case of the finite elastic deformation. It should be also mentioned here that the positive definiteness of the strain energy $A(e_{ij})$ can not be always assumed although its existence is assumed. This suggests occurrence of instability problems of the solution.

12. <u>Mixed variational formulation on the flow theory of plasticity</u>

Firstly well known Prandtl-Reuss' flow theory of plasticity must be outlined as follows:

True increment of the state vector ( $du$ , $d\sigma_{ij}$ ) of a solid must satisfy the following set of equations:

equilibrium equation : $\left(d\sigma_{ij,j}\right) = 0$ .....................(12.1)

compatibility equation : $d\varepsilon_{ij} = \frac{1}{2}\left[(\partial du_i / \partial x_j) + (\partial du_j / \partial x_i)\right]$ ...........(12.2)

boundary conditions :

geometrical B.C. $du_i = d\bar{u}_i \quad on\ S_u$ ..............(12.3)

stress B.C. $dt_i = d\sigma_{ij} n_j = d\bar{t}_i \quad on\ S_\sigma \quad (i, j = 1.2.3)$ ...........(12.4) where $S = S_u + S_\sigma$

and $du_i$ implies displacement increment.

Prandtl and Reuss presented the following equations: $d\varepsilon_{ij}^p = \sigma'_{ij} d\lambda$ ……..…..……(12.5)

where $d\lambda$ is a scalar constant of proportionality and the plastic work done $dW_p$ is given by

$$dW_p = \sigma_{ij} d\varepsilon_{ij}^p = \sigma'_{ij} \varepsilon_{ij}^p \qquad \text{…………………(12.6)}$$

and therefore,

$$dW_p = \sigma'_{ij} \sigma'_{ij} d\lambda \qquad \text{…………………(12.7)}$$

Since $dW_p \geq 0$, $d\lambda \geq 0$ *i.e.*, $\lambda$ is a positive scalar and von Mises yield condition is adopted. Therefore, denoting

$$\bar{\sigma} = \sqrt{\frac{2}{3}} \left( \sigma'_{ij} \sigma'_{ij} \right)^{1/2}, \quad d\varepsilon^p = \sqrt{\frac{2}{3}} \left( d\varepsilon_{ij}^p d\varepsilon_{ij}^p \right)^{1/2} \qquad \text{………(12.8)}$$

Mises yield condition is given by

$$\bar{\sigma} = \sqrt{3J'_2}, \quad J' = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} \qquad \text{…………………(12.9)}$$

Consequently, the following equation can be derived.

$$\bar{\sigma} = F\left( \int \bar{\sigma} d\varepsilon^p \right) \qquad \text{…………………(12.10)}$$

together, with

$$d\varepsilon_{ij}^p = \frac{3\sigma'_{ij} d\bar{\sigma}}{2\bar{\sigma}^2 F'}, \quad F' = \frac{1}{\bar{\sigma}} \frac{d\bar{\sigma}}{d\varepsilon^p} \quad \left( d\sigma \geq 0 \right) \qquad \text{………(12.11)}$$

writing eq. (12.10) as

$$\bar{\sigma} = H\left( \int d\varepsilon^p \right) \qquad \text{…………………(12.12)}$$

the following equation is obtained:

$$F' = H' / \bar{\sigma} \qquad \text{…………………(12.13)}$$

And finally the following complete set of Prandtl-Reuss equation for strain-hardening materials are derived by:

$$\left.\begin{aligned} d\varepsilon_{ij} &= \frac{(1-2\nu)}{E} d\sigma\delta_{ij} + \frac{d\sigma'_{ij}}{2G} + \frac{3\sigma'_{ij} d\bar{\sigma}}{2\bar{\sigma} H'} & d\bar{\sigma} \geq 0 \\ d\varepsilon_{ij} &= \frac{(1-2\nu)}{E} d\sigma\delta_{ij} + \frac{d\sigma'_{ij}}{2G} & d\bar{\sigma} \prec 0 \end{aligned}\right\} \qquad \text{…………………(12.14)}$$

The inverse equation of eq. (12.14) is given by

$$
\left.\begin{array}{ll}
d\sigma_{ij} = \dfrac{E}{1-2\nu} de\delta_{ij} + 2G\left[ d\varepsilon'_{ij} - \dfrac{\sigma'_{kl} d\varepsilon_{kl}}{\frac{2}{3}\bar{\sigma}^2\left(\frac{H}{3G}+1\right)}\sigma'_{ij}\right] & \sigma'_{ij} d\varepsilon_{ij} \geq 0 \\
d\sigma_{ij} = \dfrac{E}{1-2\nu} de\delta_{ij} + 2Gd\varepsilon' & \sigma'_{ij} d\varepsilon_{ij} \prec 0
\end{array}\right\} \quad \text{………(12.15)}
$$

where $de = \frac{1}{3} d\varepsilon_{ii}$

Now the problem at hand is defined by K. Washizu as follows:

Find the solution for incremental state vector ( $du$ , $d\sigma_{ij}$ ) of a given solid which satisfied:

$$
\left.\begin{array}{llll}
\textit{equilibrium equation} & : d\sigma_{ij,j} = 0 & \textit{in } V & (a) \\
\textit{compatibility equation} & : d\varepsilon_{ij} = \frac{1}{2}\left(\partial du_i / \partial x_j + \partial du_j / \partial x_i\right) & & (b) \\
\textit{boundary conditions} & : & & \\
\quad \textit{displacement B.C.} & du_i = d\bar{u}_i & \textit{on } S_u & (c) \\
\quad \textit{stress B.C.} & dt_i = d\sigma_{ij} n_j = d\bar{t}_i & \textit{on } S_\sigma & (d)
\end{array}\right\} \quad \text{…………(12.16)}
$$

Three variational principles can be proposed for the problem defined above.

(i) the principle for incremental virtual work(the upper bound theorem)

Denoting a set of the incremental displacement satisfying eq.(12.16.a) and the incremental strains by $du^*$ and $d\varepsilon^*$ respectively, the true solution of $du^*$ makes the following equation minimum:

$$
\int_v dA^* dv - \int_{s_\sigma} dt_i du^* ds \qquad \rightarrow \qquad \min
$$

where

$$
dA = \frac{3E}{2(1-2\nu)}(de)^2 + G\left[ d\varepsilon'_{ij} d\varepsilon'_{ij} - \frac{(\sigma'_{kl} d\varepsilon'_{kl})^2}{\frac{2}{3}\bar{\sigma}^2(\frac{H'}{3G}+1)}\right] \qquad \text{…………………(12.17)}
$$

and $dA^*$ is the value of $dA$ where $d\varepsilon_{ij}$ is replaced by $d\varepsilon^*_{ij}$

(ii) the principal for the incremental complementary virtual work(the lower bound theorem)

Denoting a set of the incremental stress satisfying eq.(12.16.a) and eq.(12.16.d) by $d\sigma_{ij}{}^*$, the true solution of $d\sigma_{ij}{}^*$ makes the following equation minimum:

$$
\int_v dB^* dv - \int_{s_u} dt_i^* du_i ds \qquad \rightarrow \qquad \min
$$

where

$$dB = \frac{3(1-2\nu)}{2E}(d\sigma)^2 + \frac{d\sigma'_{ij} d\sigma'_{ij}}{4G} \qquad \text{.....................(12.18)}$$

and $dB^*$ is the value of $dB$ where $d\sigma_{ij}$ is replaced by $d\sigma^*_{ij}$

(iii) the unified energy principle for the strain hardening materials obeying Prandtle-Reuss's flow theory of plasticity

For the variation of the incremental state vector($du^*, d\sigma_{ij}{}^*$) of the strain-hardening materials obeying Prandtle-Reuss's flow rule, the true solution thereof will make the following equation minimum:

$$\int_v (dA^* + dB^*)dv - \int_{s_\sigma} d\bar{t}_i du^* ds - \int_{s_u} dt_i^* d\bar{u}_i ds \quad \to \min \qquad \text{..............(12.19)}$$

13. <u>On the solution for in-plane bending of a cantilever plate due to a boundary shear given by S. P. Timoshenko</u>

S. P. Timoshenko gave a very interesting solution for in-plane bending of a cantilever plate in his celebrated textbook on the theory of elasticity as follows:

The problem is defined by the following equation of equilibrium:

$$\left.\begin{array}{l} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 \end{array}\right\} \qquad \text{..............(13.1)}$$

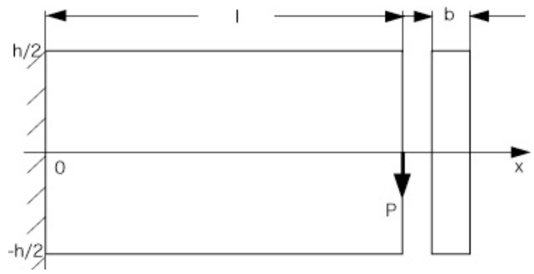



Fig.3 *In-plane bending of a cantilever plate due to a boundary shear of a palaboric distribution given by*

$$\tau_{xy} = -\frac{P}{2I}\left(\frac{h^2}{4} - y^2\right) \quad \text{when} \quad I = \frac{bh^3}{12}$$

with associated boundary conditions:

$$\left.\begin{array}{llll} x = l & \sigma_x = -\dfrac{P(l-x)y}{I} & \tau_{xy} = \dfrac{P}{2I}\left(\dfrac{h^2}{4} - y^2\right) \\ y = \pm\dfrac{h}{2} & \sigma_y\left(x, \pm\dfrac{h}{2}\right) = 0 & \tau_{xy}\left(x, \pm\dfrac{h}{2}\right) = 0 \\ x = 0 & u(0,y) = v(0,y) = 0 & \end{array}\right\}$$

...................................(13.2)

Replacing the clamped edge condition at x=0 by

$$\text{u(0,0)=v(0,0)=0,} \quad \chi(0,0) = 0 \quad \text{when } \chi = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \qquad \text{..................(13.3)}$$

The following analytical solution is derived by him:

$$\left.\begin{array}{l} u(x,y) = -\dfrac{P(l-x)^2 y}{2EI} - \dfrac{\nu P y^2}{6EI} + \dfrac{P y^3}{6IG} + \left(\dfrac{Pl^2}{2EI} - \dfrac{Ph^2}{8IG}\right)y \\ v(x,y) = \dfrac{\nu P(l-x)y^2}{2EI} + \dfrac{P(l-x)^2}{6EI} - \dfrac{Pl^2(l-x)}{2EI} + \dfrac{Pl^3}{3EI} \end{array}\right\} \qquad \text{..................(13.4)}$$

Here the approximation solution of the same problem is considered to obtain by the energy method using the following displacement functions:

$$\left.\begin{aligned} u(x,y) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^m y^n \\ v(x,y) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{mn}x^m y^n \end{aligned}\right\} \qquad \text{..................(13.5)}$$

It must be mentioned here first that the displacement functions defined by eq.(13.5) exactly satisfies the clamped edge conditions given by

$$u(0,y) = v(0,y) = 0 \qquad \text{..................(13.6)}$$

Now the approximate solution for this problem is to be searched by minimizing the following potential energy formulation $\Pi_p(a_{mn}, b_{mn})$ with respect to unknowns $(a_{mn}, b_{mn})$

$$\Pi_p(a_{mn},b_{mn}) = \frac{1}{2}\int_0^l \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{Eh}{(1-\nu^2)}\left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 + 2(1-\nu^2)\left\{\frac{1}{4}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 - (\frac{\partial u}{\partial x})(\frac{\partial v}{\partial y})\right\}\right]dxdy$$

$$-\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{Ph}{2I}(c^2-y^2)v(l,y)dy \qquad \text{..................(13.7)}$$

$$\frac{\partial \Pi_p}{\partial a_{mn}} = 0 \quad , \frac{\partial \Pi_p}{\partial b_{mn}} = 0 \qquad \text{..................(13.8)}$$

Unfortunately, however, calculation of the strain energy of a given plate does not converge.
The reason why was found that this is due to assumption of the clamped edge condition given by eq.(13.6) .

Now this clamped edge condition is replaced by the following series form:

$$\left.\begin{aligned} u(o,y) &= a_{00} + a_{01}y + a_{02}y^2 + \cdots \\ v(o,y) &= b_{00} + b_{01}y + b_{02}y^2 + \cdots \end{aligned}\right\} \qquad \text{..................(13.9)}$$

Considering $a_{00} = u(0,0) = 0, \quad b_{00} = v(0,0) = 0$ and these conditions imply no translational displacements at the origin of the coordinates. Therefore the first approximation of the clamped edge at x=0 is given by further introduction of zero rotation $\chi(x,y)$ at the origin:

$$\chi(x,y) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$i.e. \quad \chi(0,0) = \frac{1}{2}\left(a_{01} - b_{01}\right) = 0 \quad \therefore a_{01} = b_{01}$$

Thus the present analysis is reduced to Timoshenko's solution. His solution, however, is exact solution under the boundary condition of $u(0,0) = v(0,0) = \chi(0,0) = 0$ , but only the first approximate solution for $u(0,y) = v(0,y) = 0$ !!

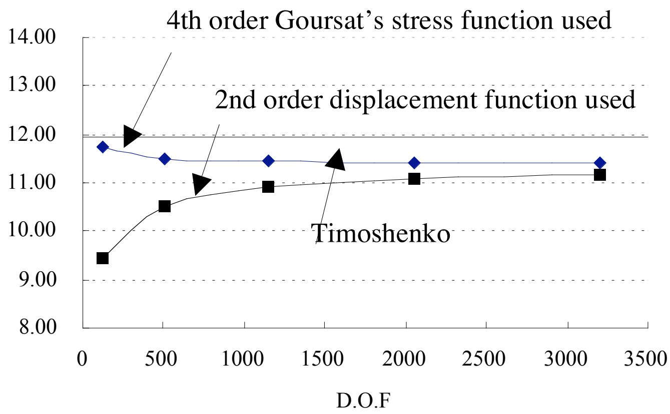
All of sudden, I recognized that the computer warned me modeling of the clamped end condition specified by $u(0,y) = v(0,y) = 0$ is impossible.

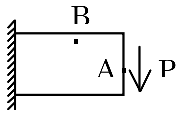
Therefore it can be concluded that this problem is not adequate to use for bench mark test of the finite elements for 2D stress analysis because it is only the first approximate solution of asymptotic convergency. Similar consideration must be made for analysis of structures like skewed bridges, sweptback wing of aircrafts and idealization of the fixed end condition in experimental study. Finite Element analysis was conducted first using the following element displacement function (NDOF=16).

$$\left.\begin{aligned} u(x,y) &= u_c - \chi_0 y + \varepsilon_{x0}x + \frac{1}{2}\gamma_{xy}y + a_1x^2 + a_2xy + a_3y^2 + a_4x^2y + a_5xy^2 \\ v(x,y) &= v_c - \chi_0 x + \varepsilon_{y0}y + \frac{1}{2}\gamma_{xy}x + b_1x^2 + b_2xy + b_3y^2 + b_4x^2y + b_5xy^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(13.10)$$

At the same time using Trefftz's method and Goursat's stress function approach, the same problem was analyzed. The results obtained are shown in Fig 4.

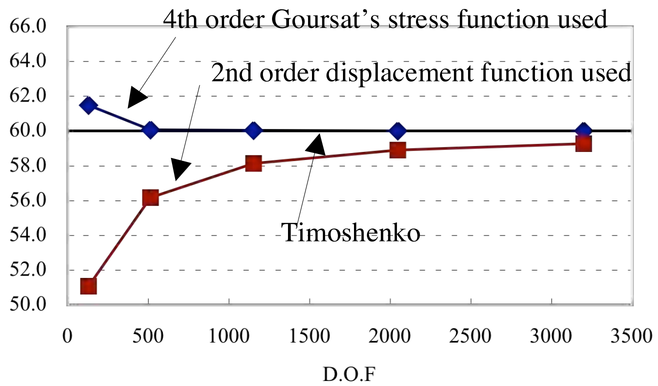
$v_A$ : vertical displacement at the point A

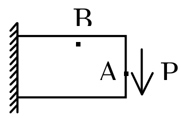



| Mesh Div. × NDOF | stress function used | displacement function used |
|---|---|---|
| 4×2×16 | 11.7195 | 9.4399 |
| 8×4×16 | 11.4996 | 10.5163 |
| 12×6×16 | 11.4347 | 10.9196 |
| 16×8×16 | 11.4063 | 11.0912 |
| 20×10×16 | 11.3909 | 11.1780 |

$(\sigma_x)_B$ : stress at the point B




| Mesh Div. × NDOF | stress function used | displacement function used |
|---|---|---|
| 4×2×16 | 61.4766 | 51.0777 |
| 8×4×16 | 60.0641 | 56.1607 |
| 12×6×16 | 60.0287 | 58.1254 |
| 16×8×16 | 60.0138 | 58.8946 |
| 20×10×16 | 60.0071 | 59.2698 |

Figure 4: Inplane bending analysis of a cantilever plate subjected to a boundary shear of parabolic distribution (divided by square mesh

A solution obtained using Modified Reissner method with the element displacement function defined by eq. (13.11) is shown by curve –■– , while the other using the equilibrium displacement functions is shown by the curve –◆– in this figure. Fig 4 also shows the convergency characteristics of the calculated displacement $v_A$ and stress $\sigma_A$ respectively. It can be seen that the curve –■– gives always the upper bound solution for both $v_A$ and $\sigma_A$, while the other curve –◆– gives the lower bound solution.

14. Concluding remark:

So far I have discussed a new mixed variational method developed which can be applied to the material and geometrical nonlinear problems of solids separately. Development of the method is underway for their coupled problems using standard incremental procedure. Application of the present method to the problems of solid contact, microplar materials etc. area also being considered,

Finally it may be the last problem for not only researchers but also practisionors to challenge the mixed variational formulation of multiphysics problems by finding another variational principle for enthropy conservation law governing the general transport phenomena.

As the concluding remark, I should be mentioned that already 130 years ago Josiah Willard Gibbs, the founder of the statistical thermodynamics, predicted that both minimum principles of strain energy and complementary energy would be assured if existence and positive definiteness of $A(\varepsilon_{ij})$ and $B(\sigma_{ij})$ can be assumed.

Therefore I believe that he might have foreseen existence of the unified energy principle I discussed in this paper.